The Riemann Hypothesis

Introduction

The aim of the proof is first to demonstrate that selecting N random numbers, as shown by Reuben and Hersh[1], that there is an equal chance of selecting a number with an odd or even number of factors. Then the aim is to show that this also applies from 1 to N.

(1) shows a well known equation. It is desired to transform (1) so it is represented in a matrix of rows and columns, but with positive and negative terms as in (4) instead of fractions as in (1). If the matrix is structured correctly it should be mathematically equivalent to (1) and then the ratio of the positive to negative terms can be deduced both for N random numbers and 1 to N.

In a further aim the construction of the matrix separates the nonsquare terms in (4) from numbers that are multiples of these terms. These multiples contain squares, and represent the zeroes of the Mobius Function. Initially these will also be represented as positive and negative terms according to whether they have an even or odd number of factors.

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[1] *Davis, P.J., and R. Hersch. 1981. The Mathematical Experience*

The next aim is to show that there is an equal chance of selecting a positive or negative term containing squares, and then the nonsquare terms in the matrix from (4). Then that this applies from 1 to N as well.

The next aim is to convert all the terms in the matrix and tables to terms in the Mobius Function by changing z to equal 0 instead of 1. This gives an exact analogy to the Mobius Function so the results prove the Riemann Hypothesis.

Similar to in the attached article[2], N random numbers are selected and placed in parts of the matrix. The random numbers selected are defined as only the inverses of the integers. As in the Mobius Function, if any term has an odd number of factors it is defined as negative and if it has an even number of factors it is defined as positive. Here, if a randomly selected number contains a square it is still defined as positive or negative rather than zero. Later the terms containing squares become zeroes.

Proposition X refers to Lemma X. For example Proposition 2 refers to Lemma 2.

Proofs

Proposition 1: To prove that

(p is prime) $\prod_{n=1}^{\infty} \frac{p-1}{p} = \frac{1}{2}\frac{2}{3}\frac{4}{5}\frac{6}{7}...$ through all the primes in ascending order = 0. (1)

[2] *Davis, P.J., and R. Hersch. 1981. The Mathematical Experience*

Proposition 2: That is there are an infinite number of primes.

Lemma 1: Consider a ruler of infinite length and equally spaced markings, like markings on a normal ruler. Cross off every second marking, this leaves $\frac{1}{2}$ of the ruler remaining. Cross off every third marking, this leaves $\frac{2}{3}$ of the ruler remaining. Continue doing this through the primes in ascending order.

With an infinite number of primes there can be no part of the ruler not crossed off hence the fraction of the ruler remaining equals zero, therefore (1) = 0. Q.E.D.

Lemma 2: If there are a finite number of primes then (1) equals a fraction that is nonzero, $\frac{x}{y}$ with $1-\frac{x}{y}$ of the ruler remaining.

This is impossible since the ruler markings are either primes or have the primes as factors. The remaining part of the ruler must represent at least one more prime or composite number, so the number of primes is infinite. Q.E.D.

(1) can also be written as:

$$(1-\frac{1}{2^z})(1-\frac{1}{3^z})(1-\frac{1}{5^z})... \qquad (2)$$

where z=1 through all the primes in ascending order. When z=2 this equals $\frac{6}{\pi^2}$.

When z=1:

$$(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})... \text{ through all the primes in ascending order.} \qquad (3)$$

Proposition 3: (3) is equivalent to (1) and equals 0.

Lemma 3: For example $\frac{4}{5}=(1-\frac{1}{5})$, $\frac{1}{2}=(1-\frac{1}{2})$, $\frac{10}{11}=(1-\frac{1}{11})$, etc. Q.E.D.

Corollary: (2) can be expanded as

$$1-\frac{1}{2^z}-\frac{1}{3^z}-\frac{1}{5^z}+\frac{1}{6^z}-\frac{1}{7^z}-\frac{1}{11^z}... \qquad (4)$$

Proposition 4: With z=1, (4)=(1)=0 if the terms of (4) are in the correct order.

Lemma 4: The terms in (4) are an expansion of (3) so some correct order must exist so (4)=(1). Q.E.D.

Definition of matrix: The matrix is a structure of rows and columns in which the terms of (4) are arranged. Each term from (4) has an associated table in which multiples of that term are placed. Since the terms in the matrix contain no squares all the terms in the tables contain squares.

Definition of tick: If a particular term from (4) has been selected from N random numbers or 1 to N a tick is placed next to that term in the matrix. If a term from (4) has not been selected yet then there is no tick. The aim is that for example if N random numbers are selected then some of the terms from (4) will be selected before others.

By representing terms that have been selected with a tick then we can see at any time how many of these terms have been selected so far. This also separates the nonsquare numbers from those that contain a square, which are placed in tables associated with the terms.

Definition of number: The numbers selected from N random numbers or 1 to N are the inverses of the integers. Some of these will be the terms in (4), and others will be multiples of those terms.

Definition of terms: Terms are numbers from (4) or multiples of those terms.

Definition of square and nonsquare: Numbers containing squares would be placed in tables. Numbers containing no squares are from (4) and when they are selected they are not placed in any table, but a tick is placed next to that number in the matrix. As the matrix is populated with numbers the nonsquare terms are thus separated from the terms containing squares.

Definition of positive and negative terms: Positive terms have an odd number of factors and negative terms have an even number of factors. In (4) this is a consequence of the expansion of (3). Numbers containing squares placed in tables are set as positive or negative in the same way.

Definition of table: Each and every nonsquare term from (4) in the matrix has an associated table, which is unique to that term. It contains only numbers that are multiples of that term. If all numbers were randomly selected then all the terms in the matrix would be ticked and all the tables would contain all the multiples of the term they are associated with.

A number selected can be placed in more than one table. For example $+\frac{1}{72}$ might go into the tables for $-\frac{1}{2}$, $-\frac{1}{3}$, and $+\frac{1}{6}$. In another example $+\frac{1}{4}$ would go into the table for $-\frac{1}{2}$ because it is a multiple of $-\frac{1}{2}$. $-\frac{1}{63}$ would go into the associated tables of $-\frac{1}{3}$, $-\frac{1}{7}$ and $+\frac{1}{21}$ because it is a multiple of all these terms. A copy of all randomly selected numbers would be placed in the table associated with 1.

This is because the positive and negative terms in their position in the matrix and tables are analogous to crossing the numbers off on the ruler, and a number for example 72 is crossed off by 2, 3 and 6.

For example on the ruler the twos and threes are crossed off and these intersect at the 6's. If we select random numbers we find half have a factor of two and one third have a factor of 3, and this implies that for example $\frac{1}{24}$ have 24 as a factor. In the matrix we see this as positive and negative terms like in the zeta function rather than as fraction like in (1). So when a multiple of 6 is selected randomly it is placed in the table of $+\frac{1}{6}$, $-\frac{1}{2}$, and $-\frac{1}{3}$. If for example $+\frac{1}{150}$ was randomly selected, it would be placed in the tables of $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{5}$, $+\frac{1}{6}$, $+\frac{1}{10}$ $+\frac{1}{15}$, just as the numbers would be crossed off on the ruler.

One aim of this is to determine the ratio of positive to negative terms, and whether the randomly selected terms are equally likely to be positive or negative.

Definition of rows: The numbering of the rows starts at zero so the row number is the same as the number of factors in the terms in that row. In the Zeroth row there is 1 by itself. It has a corresponding table and all numbers as they are randomly selected are also placed in this table, and then other tables according to their factors. 1 is the only term that contains all numbers selected in its table.

$-\frac{1}{2}$'s table would contain all the numbers selected that had 2 as a factor, $-\frac{1}{3}$'s table would contain all the numbers selected that had three as a factor, etc. Since for example we would expect there to be about half as many numbers in $-\frac{1}{2}$'s table as 1's table then this is equivalent to the equation $1-\frac{1}{2}$ or $\frac{1}{2}$. Since we would expect there to be about $\frac{1}{3}$ as many numbers in $-\frac{1}{3}$'s table as in 1's table, this is equivalent to $1-\frac{1}{3}$ or $\frac{2}{3}$.

In the first row are all the terms from (4) with one prime, for example $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{5}$, $-\frac{1}{7}$, $\frac{1}{11}$… Generally there would be about $\frac{1}{p}$ as many numbers in $-\frac{1}{p}$'s table compared to 1's table, which contains all numbers selected.

In the second row go all the terms from (4) which have two primes multiplied together, in the third row those terms from (4) with three primes multiplied together, and generally in the Nth row those terms from (4) with N primes multiplied together.

p_1, p_2, p_3,… are defined as unique primes. In the second row a term $+\frac{1}{p_1 p_2}$'s table would contain about $\frac{1}{p_1 p_2}$ as many numbers as 1's table as N goes to infinity.

For example:

Zeroth Row: 1(Table) (5)

First Row: $-\frac{1}{2}$(Table), $-\frac{1}{3}$ (Table), $-\frac{1}{5}$ -1/5(Table)… (6)

Second Row: $+\frac{1}{6}$ (Table), $+\frac{1}{10}$ (Table), $+\frac{1}{15}$ (Table)… (7)

Corollary: Each Nth row contains all possible terms that have N unique primes multiplied together. For example the first row contains each example of -1/p where p is a prime, so the inverses of all the primes are there.

Definition of groups: A group is part or all of the terms formed from multiplying together a finite number of fractions from (1) and expanding it. For example $\frac{1}{2}\frac{2}{3}=(1-\frac{1}{2})(1-\frac{1}{3})=(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{6})$. So $(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{6})$ would be a group.

The term $+\frac{1}{6}$ would be in row 2 because it contains no squares and has two prime factors, and it is associated in a group with 1, $-\frac{1}{2}$ and $-\frac{1}{3}$. This group has terms on the Zeroth row as 1 , the first row as $-\frac{1}{2}$ and $-\frac{1}{3}$ and the second row as $+\frac{1}{6}$.

In another example $\frac{4}{5}\frac{6}{7} = (1-\frac{1}{5})(1-\frac{1}{7}) = (1-\frac{1}{5}-\frac{1}{7}+\frac{1}{35})$ so 1 is in the Zeroth row, $-\frac{1}{5}$ and $-\frac{1}{7}$ in the first row and $+\frac{1}{35}$ in the second row. $(1-\frac{1}{5}-\frac{1}{7}+\frac{1}{35})$ is a group. Groups, for example $(1-\frac{1}{5}-\frac{1}{7}+\frac{1}{35})$, are defined as having an equal number of positive and negative terms.

If N random numbers are selected, which are the inverses of the integers; we can start to populate the tables of each term. As the tables that are associated with a group are populated we should see the ratio of positive to negative terms in those tables change.

Proposition 5: The ratio of positive to negative terms in the tables of a group ratio is 1:1 as N goes to infinity.

Lemma 5: For example on the average, double each of the terms of a group would be selected, three times each term in a group, etc and placed in the associated tables. The third line would generally have 1/3 the chance of being selected as line 1.

$$\begin{array}{llll} 1 & -\frac{1}{2} & -\frac{1}{3} & +\frac{1}{6} \\ & +\frac{1}{4} & +\frac{1}{6} & -\frac{1}{12} \\ & +\frac{1}{6} & +\frac{1}{9} & -\frac{1}{18} \end{array} \quad (8)$$

The terms change in rows between positive and negative. For example $+\frac{1}{6}$ becomes $-\frac{1}{12}$ and $-\frac{1}{18}$, etc. Even if the ratio of positive and negative terms was not 1:1 then this alternation would make the overall ratio 1:1. Q.E.D.

Corollary: This applies to all groups with two primes p_1 and p_2, with 4 terms:

$$(1-\frac{1}{p_1})(1-\frac{1}{p_2}) = (1-\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_1 p_2}) \qquad (9)$$

As random numbers are placed in the tables of a group like (9), each will have a ratio of positive to negative terms of 1:1 as N goes to infinity. Multiples of these terms will all balance out with positive and negative terms so adding the tables in the group of 1, $-\frac{1}{p_1}$, $-\frac{1}{p_2}$, and $+\frac{1}{p_1 p_2}$ generally will go to zero as N random numbers goes to infinity.

Proposition 6: Groups can defined for all nonsquare terms so that they contain equal numbers of positive and negative terms.

Lemma 6: Next is to construct all groups that include terms in the third row and show these too have an equal number of positive and negative terms. As an example

$$\frac{1}{5}(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{6}) \tag{10}$$

This gives

$$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}-\frac{1}{30} \tag{11}$$

So this adds 4 new terms to

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{6} \tag{12}$$

which are

$$-\frac{1}{5}+\frac{1}{10}+\frac{1}{15}-\frac{1}{30} \tag{13}$$

Again there are an equal number of positive and negative terms. $(-\frac{1}{5}+\frac{1}{10}+\frac{1}{15}-\frac{1}{30})$ is defined as a separate group to $(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{6})$. As the tables associated with the terms of (13) are populated, there are an equal number of positive and negative terms as N goes to infinity.

We can do the same generally like in (10) which completely populates the third line with all possible inverses of numbers with three different factors.

Doing this creates all possible groups of 4 terms like (13) and as the tables of each are populated each goes to zero as N goes to infinity.

With the same method:

$$\frac{1}{7}(1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}-\frac{1}{30}) \qquad (14)$$

which equals

$$1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{10}+\frac{1}{14}+\frac{1}{15}+\frac{1}{21}-\frac{1}{30}+\frac{1}{35}-\frac{1}{42}-\frac{1}{70}-\frac{1}{105}+\frac{1}{210} \qquad (15)$$

This gives 8 extra terms as another group with an equal number of positive and negative terms.

$$(-\frac{1}{7}+\frac{1}{14}+\frac{1}{21}+\frac{1}{35}-\frac{1}{42}-\frac{1}{70}-\frac{1}{105}+\frac{1}{210}) \qquad (16)$$

According to this method all possible combinations are done so all possible 8 termed groups like this are created. The ratio of positive to negative numbers in their associated tables is 1:1 as N goes to infinity. So the process of duplicating (1) with positive and negative terms instead of fractions continues. More terms are created so all possible terms in (4) are assigned to various groups. Each of these groups would have an equal number of positive and negative terms and so each group equals zero as N goes to infinity. Q.E.D.

Proposition 7: Selecting randomly from all tables in the matrix there is an equal chance of selecting a positive or negative term.

Lemma 7: Each unique group's positive and negative terms can be considered as like a unique coin, with positive for heads and negative for tails. Since there are an infinite number of groups there would be an infinite number of coins. If an infinite number of coins are tossed in this way, the overall ratio of heads to tails would still be 1:1. Q.E.D.

Proposition 8: As N random numbers are selected according to the method of Davis and Hersh, there is an equal chance of selecting a number with an odd or even number of factors.

Lemma 8: The numbers in the tables, and the terms in the matrix with a tick beside them are the same as the N random numbers selected. A positive term has an odd number of factors and a negative term has an even number of factors. Q.E.D.

Proposition 9: There is an equal chance of selecting a positive or negative term from the tables, and also separately from the nonsquare terms.

Lemma 9: The tables are defined as containing all the terms with squares in them, and the terms outside the tables in the matrix are defined as containing no squares. The distribution of the nonsquare terms is the same as for the terms in their associated tables therefore there is an equal chance of selecting a positive or negative term out of the non square terms. Q.E.D.

Proposition 10: The terms in the tables represent the zeroes of the Mobius Function.

Lemma 10: As Davis and Hersh explain, terms containing squares are set as neither positive nor negative but as zero. Q.E.D.

Proposition 11: Each term in the matrix and associated tables is assumed to have an exponent of z with its denominator. Since z=1 in the matrix so far this is not shown. Setting z to equal zero converts the matrix into representing the Mobius Function.

Lemma 11: The matrix can be converted into the Mobius Function by making z=0.

When z=1

$$1^z - \frac{1}{2^z} - \frac{1}{3^z} - \frac{1}{5^z} + \frac{1}{6^z} \ldots = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} \ldots \quad (17)$$

And for z=0

$$1^0 - \frac{1}{2^0} - \frac{1}{3^0} - \frac{1}{5^0} + \frac{1}{6^0} \ldots = 1 - 1 - 1 - 1 + 1 \ldots \qquad (18)$$

The terms in the matrix then become +1 or -1, and the terms in the tables can be set to each equal zero. The terms in the matrix remain in the same positions and the randomly selected numbers are as before, but for z=0. Each group equals zero and the tables associated with each group sums to zero. Q.E.D.

Proposition 12: That the terms in the matrix, shown as (18) have the required rate of growth.

Lemma 12: As Church and Goodhouse[3] show with Hausdorff's Inequality, if there is an equal chance of selecting a nonsquare term which is positive or negative then with probability 1:

M(N) grows no faster than $ON^{\frac{1}{2}+\epsilon}$ as N goes to infinity (19)

for N random numbers. So with N random numbers instead of 1 to N the Riemann Hypothesis is true. Q.E.D.

Proposition 13: That the integers in ascending order from 1 to N could be selected from N random numbers as N goes to infinity. Assuming the numbers selected to populate the

[3] I.J. Good and R.F. Churchhouse, "The Riemann hypothesis and pseudorandom features of the Mobius sequence", *Mathematics of Computation* **22** (1968) 857-864.

matrix are random, is it possible to select the integers in order, or would this not be possible to occur randomly?

Lemma 13: (1) can apply to both 1 to N and N random numbers, as Davis and Hersh show. By marking off every second number on a ruler we show that half the integers have 2 as a factor. By selecting N random numbers we can show half of those selected have two as a factor,

In the same way marking off every third number on the ruler is the same as selecting N random numbers with $\frac{1}{3}$ of them having 3 as a factor. The random numbers have $\frac{1}{2}$ of them with 2 as a factor, $\frac{1}{3}$ with 3 as a factor and generally $\frac{1}{p}$ of them with p as a factor, p is any prime. This is the same for the numbers from 1 to N.

For example if we tossed a coin an infinite number of times and the ratio of Heads to Tails was not 1:1 we would define this as a proof the toss was not random. If an infinite number of terms were selected and $\frac{1}{p}$ of them did not contain p as a factor this would be a definition of the number selection not being random.

Since for 1 to N each factor p occurs $\frac{1}{p}$ of the time and this is the same in N random numbers then there is no test of randomness that would say 1 to N is not a random selection. Therefore 1 to N could be selected randomly as N goes to infinity. Q.E.D.

Proposition 14: That the tables associated with each group in the matrix would have a ratio of positive and negative terms of 1:1, selecting the numbers from 1 to N in ascending order as N goes to infinity.

Lemma 14: As shown in (11) a group would have terms populate its tables in ascending order periodically. For example $+\frac{1}{4}$ is placed, then $+\frac{1}{6}$ then $-\frac{1}{12}$ bringing that row to zero. Then $+\frac{1}{6}$, $-\frac{1}{18}$, and $+\frac{1}{36}$. In this example $+\frac{1}{6}$ can be left in the table for clarity. It only occurs an even and finite number of times.

If 1 is not considered then in this group there are alternating ratios of 1:3 and 2:3 of positive terms so overall this will be a ratio of 1:1 as N goes to infinity. This applies for all groups except for those like (13) and (20) where the group's tables are populated as equal numbers of positive and negative terms.

$$1 \qquad -\frac{1}{2} \qquad -\frac{1}{3} \qquad +\frac{1}{6} \tag{20}$$

$$+\frac{1}{4} \qquad +\frac{1}{6} \qquad -\frac{1}{12} \tag{21}$$

$$+\frac{1}{6} \qquad +\frac{1}{9} \qquad -\frac{1}{18} \tag{22}$$

The terms in each group's tables over and over have an equal ratio of positive and negative terms, as with N random numbers. Q.E.D.

Proposition 15: If the numbers from 1 to N in ascending order are possible to select randomly as N goes to infinity then Hausdorff's Inequality applies to 1 to N.

Lemma 15: This is explained by Davis and Hersh.

Theorem: M(N) grows no faster than

$ON^{\frac{1}{2}+\epsilon}$ as N goes to infinity for 1 to N

Therefore the Riemann hypothesis is true.

Also should be true for any series of random numbers, and any seres of random numbers should have twin and other primes.